MATHEMATICS FOR THE NONMATHEMATICIAN

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LOGIC AND MATHEMATICS

Geometry will draw the soul toward truth and create the spirit of philosophy.

PLATO

3-1 INTRODUCTION

Mathematics has its own ways of establishing knowledge, and the understanding of mathematics is considerably promoted if one learns first just what those ways are. In this chapter we shall study the concepts which mathematics treats; the method, called deductive proof, by which mathematics establishes its conclusions; and the principles or axioms on which mathematics rests. Study of the contents and logical structure of mathematics leaves untouched the subject of how the mathematician knows what conclusions to establish and how to prove them. We shall therefore present a brief and preliminary discussion of the creation of mathematics. This topic will recur as we examine the subject matter itself in subsequent chapters.

Since mathematics, as we conceive the subject today, was fashioned by the Greeks, we shall also attempt to see what features of Greek thought and culture caused these people to remodel what the Egyptians and Babylonians had pursued for several thousand years.

3-2 THE CONCEPTS OF MATHEMATICS

The first major step which the Greeks made was to insist that mathematics must deal with abstract concepts. Let us see just what this means. When we first learn about numbers we are taught to think about collections of particular objects such as two apples, three men, and so on. Gradually and rather subconsciously we begin to think about the numbers 2, 3, and other whole numbers without having to associate them with physical objects. We soon reach the more advanced stage of adding, subtracting, and performing other operations with numbers without having to handle collections of objects in order to understand these operations or to see that the results agree with experience. Thus we soon become convinced that 4 times 5 must be 20, whether these numbers represent quantities of apples, horses, or even purely

imaginary objects. By this time we are really dealing with concepts or ideas, for the whole numbers do not exist in nature. Any whole number is rather an abstraction of a property which is common to many different collections or sets of objects.

The whole numbers then are ideas, and the same is true of fractions such as $\frac{2}{3}$, $\frac{5}{7}$, and so on. In the latter case, too, the formulation of the physical relationship of a part of an object to the whole, whether it refers to pies, bushels of wheat, or to a smaller monetary value in relation to a larger one, again leads to an abstraction. Mathematicians formulate operations with fractions, that is, combining parts of an object, taking one part away from the other, or taking a part of a part, in such a way that the result of any operation on abstract fractions agrees with the corresponding physical occurrence. Thus the mathematical process of, say adding $\frac{2}{3}$ and $\frac{4}{5}$, which yields $\frac{22}{15}$, expresses the addition of $\frac{2}{3}$ of a pie and $\frac{4}{5}$ of a pie, and the result tells us how many parts of a pie one would actually have.

Whole numbers, fractions, and the various operations with whole numbers and fractions are abstractions. Although this fact is rather easy to understand, we tend to lose sight of it and cause ourselves unnecessary confusion. Let us consider an example. A man goes into a shoe store and buys 3 pairs of shoes at 10 dollars per pair. The storekeeper reasons that 3 pairs times 10 dollars is 30 dollars and asks for 30 dollars in return for the 3 pairs of shoes. If this reasoning is correct, then it is equally correct for the customer to argue that 3 pairs times 10 dollars is 30 pairs of shoes and to walk out with 30 pairs of shoes without handing the storekeeper one cent. The customer may end up in jail, but he may console himself while he languishes there that his reasoning is as sound as the storekeeper's.

The source of the difficulty is, of course, that one cannot multiply shoes by dollars. One can multiply the number 3 by the number 10 and obtain the number 30. The practical and no doubt obligatory physical interpretation of the answer in the above situation is that one must pay 30 dollars rather than walk out with 30 pairs of shoes. We see, therefore, that one must distinguish between the purely mathematical operation of multiplying 3 by 10 and the physical objects with which these numbers may be associated.

The same point is involved in a slightly different situation. Mathematically $\frac{2}{1}$ is equal to $\frac{4}{2}$. But the corresponding physical fact may not be true. One may be willing to accept 4 half-pies instead of 2 whole pies, but no woman would accept 4 half-dresses in place of 2 dresses or 4 half-shoes in place of 1 pair of whole shoes.

The Egyptians and Babylonians did reach the stage of working with pure numbers dissociated from physical objects. But like young children of our civilization, they hardly recognized that they were dealing with abstract entities. By contrast, the Greeks not only recognized numbers as ideas but emphasized that this is the way we must regard them. The Greek philosopher

Plato, who lived from 428 to 348 B.C. and whose ideas are representative of the classical Greek period, says in his famous work, the *Republic*,

> *We must endeavor that those who are to be the principal men of our State go and learn arithmetic, not as amateurs, but they must carry on the study until they see the nature of numbers with the mind only; . . . arithmetic has a very great and elevating effect, compelling the soul to reason about abstract number, and rebelling against the introduction of visible or tangible objects into the argument.*

The Greeks not only emphasized the distinction between pure numbers and the physical applications of such numbers, but they preferred the former to the latter. The study of the properties of pure numbers, which they called *arithmetica*, was esteemed as a worthy activity of the mind, whereas the use of numbers in practical applications, *logistica*, was deprecated as a mere skill.

Geometrical thinking prior to the classical Greek period was even less advanced than thinking about numbers. To the Egyptians and Babylonians the words "straight line" meant no more than a stretched rope or a line traced in sand, and a rectangle was a piece of land of a particular shape. The Greeks began the practice of treating point, line, triangle, and other geometrical notions as concepts. They did of course appreciate that these mental notions are suggested by physical objects, but they stressed that the concepts differ from the physical examples as sharply as the concept of time differs from the passage of the sun across the sky. The stretched string is a physical object illustrating the concept of line, but the mathematical line has no thickness, no color, no molecular structure, and no tension.

The Greeks were explicit in asserting that geometry deals with abstractions. Speaking of mathematicians, Plato says,

> *And do you not know also that although they make use of the visible forms and reason about them, they are thinking not of these, but of the ideals which they resemble; not of the figures which they draw, but of the absolute square and the absolute diameter . . . they are really seeking to behold the things themselves, which can be seen only with the eye of the mind?*

On the basis of elementary abstractions, mathematics creates others which are even more remote from anything real. Negative numbers, equations involving unknowns, formulas, and other concepts we shall encounter are abstractions built upon abstractions. Fortunately, every abstraction is ultimately derived from, and therefore understandable in terms of, intuitively meaningful objects or phenomena. The mind does play its part in the creation of mathematical concepts, but the mind does not function independently of the outside world. Indeed the mathematician who treats concepts that have no physically real or intuitive origins is almost surely talking nonsense. The intimate connection between mathematics and objects and events in the physical world is

reassuring, for it means that we can not only hope to understand the mathematics proper, but also expect physically meaningful and valuable conclusions.

The use of abstractions is not peculiar to mathematics. The concepts of force, mass, and energy, which are studied in physics, are abstractions from real phenomena. The concept of wealth, an abstraction from material possessions such as land, buildings, and jewelry, is studied in economics. The concepts of liberty, justice, and democracy are familiar in political science. Indeed, with respect to the use of abstract concepts, the distinction between mathematics on the one hand and the physical and social sciences on the other is not a sharp one. In fact, the influence of mathematics and mathematical ways of thinking on the physical sciences especially has led to ever increasing use of abstract concepts including some, as we shall see, which may have no direct real counterpart at all, any more than a mathematical formula has a direct real counterpart.

The very fact that other studies also engage in abstractions raises an important question. Mathematics is confined to some abstractions, numbers and geometrical forms, and to concepts built upon these basic ones. Abstractions such as mass, force, and energy belong to physics, and still other abstractions belong to other subjects. Why doesn't mathematics also treat forces, wealth, and justice? Certainly these concepts are also worthy of study. Did the mathematicians make an agreement with physicists, economists, and others to divide the concepts among themselves? The restriction of mathematics to numbers and geometrical forms is partly a historical accident and partly a deliberate decision made by the Greeks. Numbers and geometrical forms had already been introduced by the Egyptians and Babylonians, and their utility in daily life was established. Since the Greeks learned the rudiments of mathematics from these civilizations, the sheer weight of tradition might have caused them to continue the practice of regarding mathematics as the study of numbers and geometrical figures. But people as original and bold in thought as the Greeks would not have been bound merely by tradition, had they not found in numbers and geometrical forms sharp and clear notions which appealed to their delight in the processes of exact thinking. However, an even more compelling reason was their belief that numerical and geometrical properties and relationships were basic, that they underlay the phenomena of the physical world and the design of the entire universe. Hence to understand the world one should seek this mathematical essence. The brilliance and depth of their conception of the universe will be revealed more and more as we proceed.

When one compares the pre-Greek and Greek understanding of the concepts of mathematics and notes the sharp transition from the concrete to the abstract, another question presents itself. The Greeks eliminated the physical substance and retained only the idea. Why did they do it? Surely it is more difficult to think about abstractions than about concrete things. Also it would

seem that an attempt to study nature by concentrating on just a few aspects of physical objects rather than on the objects themselves would fall far short of effectiveness.

Insofar as the emphasis on abstractions is concerned, the Greeks saw at once what any thinking people would see sooner or later. One advantage of treating abstractions is the gain in generality. When a child learns that $5 + 5 = 10$, he acquires in one swoop a fact which applies to hundreds of situations. Likewise a theorem proved about the abstract triangle applies to a triangular piece of land, a musical percussion instrument, and a triangle determined by three heavenly bodies at any instant of time. It has been said that the process of abstraction amounts to giving the same name to different things, but this very recognition that different objects possess the common property named in the abstraction carries with it the implication that anything true of the abstraction will apply to the several objects. Part of the secret of the power of mathematics is that it deals with abstractions.

Another advantage of abstraction was also clear to the Greeks. Abstracting from a physical situation just those properties which are to be studied frees the mind from burdensome and irrelevant details and enables one to concentrate on the features of interest. When one wishes to determine the area of a piece of land, only shape and size are relevant, and it is desirable to think only about these and not about the fertility of the soil.

The emphasis on mathematical abstractions by the classical Greeks was part and parcel of their outlook on the entire universe. They were concerned with truths, and leading philosophical schools, notably the Pythagoreans and the Platonists, maintained that truths could be established only about abstractions. Let us follow their argument. The physical world presents various objects to the senses. But the impressions received by the senses are inexact, transitory, and constantly changing; indeed, the senses may be even deceived, as by mirages. However, truth, by its very meaning, must consist of permanent, unchanging, definite entities and relationships. Fortunately, the intelligence of man excited to reflection by the impressions of sensible objects may rise to higher conceptions of the realities faintly exhibited to the senses, and so man may rise to the contemplation of ideas. These are eternal realities and the true goal of thought, whereas mere "things are the shadows of ideas thrown on the screen of experience."

Thus Plato would say that there is nothing real in a horse, a house, or a beautiful woman. The reality is in the universal type or idea of a horse, a home, or a woman. The ideas, among which Plato emphasized Beauty, Justice, Intelligence, Goodness, Perfection, and the State, are independent of the superficial appearances of things, of the flux of life, and of the biases and warped desires of man; they are in fact constant and invariable, and knowledge concerning them is firm and indestructible. Real and eternal knowledge concerns these ideas, rather than sensuous objects. This distinction between the intelligible world and the world revealed by the senses is all-important in Plato.

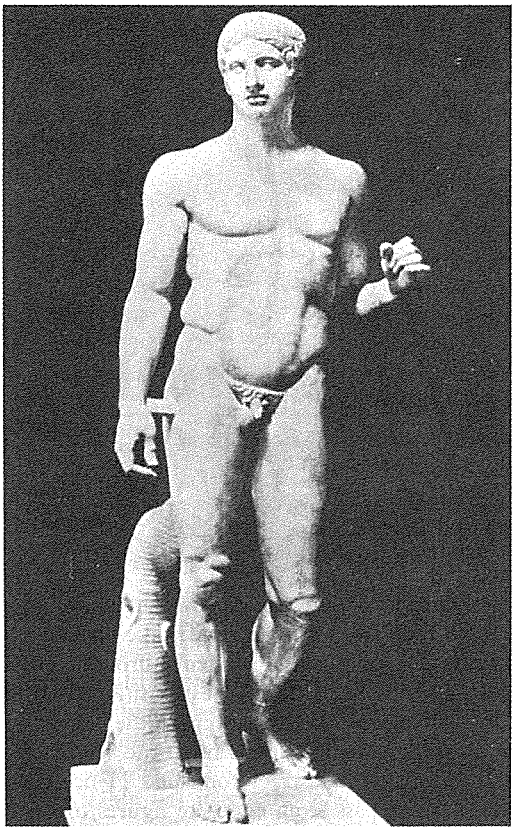

Fig. 3–1.
Polyclitus: *Spear-bearer* (*Daryphorus*). National Museum, Naples.

To put Plato's doctrine in everyday language, fundamental knowledge does not concern itself with what John ate, Mary heard, or William felt. Knowledge must rise above individuals and particular objects and tell us about broad classes of objects and about man as a whole. True knowledge must therefore of necessity concern abstractions. Plato admits that physical or sensible objects suggest the ideas just as diagrams of geometry suggest abstract geometrical concepts. Hence there is a point to studying physical objects, but one must not lose himself in trivial and confusing minutiae.

The abstractions of mathematics possessed a special importance for the Greeks. The philosophers pointed out that, to pass from a knowledge of the world of matter to the world of ideas, man must train his mind to grasp the

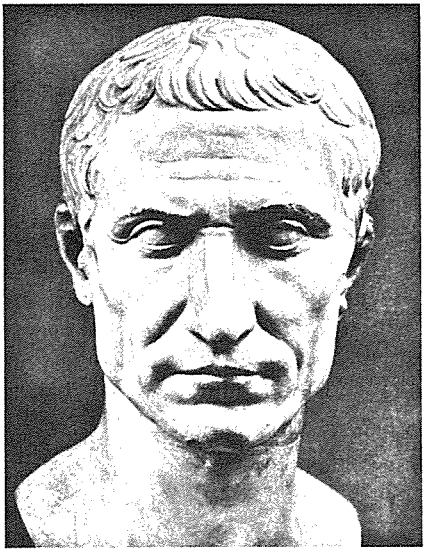

FIG. 3–2.
Bust of Caesar. Vatican.

ideas. These highest realities blind the person who is not prepared to contemplate them. He is, to use Plato's famous simile, like one who lives continuously in the deep shadows of a cave and is suddenly brought out into the sunlight. The study of mathematics helps make the transition from darkness to light. Mathematics is in fact ideally suited to prepare the mind for higher forms of thought because on the one hand it pertains to the world of visible things and on the other hand it deals with abstract concepts. Hence through the study of mathematics man learns to pass from concrete figures to abstract forms; moreover, this study purifies the mind by drawing it away from the contemplation of the sensible and perishable and leading it to the eternal ideas. These latter abstractions are on the same mental level as the concepts of mathematics. Thus, Socrates says, "The understanding of mathematics is necessary for a sound grasp of ethics."

To sum up Plato's position we may say that while a little knowledge of geometry and calculation suffices for practical needs, the higher and more advanced portions tend to lift the mind above mundane considerations and enable it to apprehend the final aim of philosophy, the idea of the Good. Mathematics, then, is the best preparation for philosophy. For this reason Plato recommended that the future rulers, who were to be philosopher-kings, be trained for ten years, from age 20 to 30, in the study of the exact sciences, arithmetic, plane geometry, solid geometry, astronomy, and harmonics (music). The oft-repeated inscription over the doors of Plato's Academy, stating that no one ignorant of mathematics should enter, fully expresses the importance

Fig. 3–3. *Parthenon,* Athens.

he attached to the subject, although modern critics of Plato read into these words his admission that one would not be able to learn it after entering. This value of mathematical training led one historian to remark, "Mathematics considered as a science owes its origins to the idealistic needs of the Greek philosophers, and not as fable has it, to the practical demands of Egyptian economics."

The preference of the Greeks for abstractions is equally evident in the art of the great sculptors, Polyclitus, Praxiteles, and Phidias. One has only to glance at the face in Fig. 3–1 to observe that Greek sculpture of the classical period dwelt not on particular men and women but on types, ideal types. Idealization extended to standardization of the ratios of the parts of the body to each other. Polyclitus believed, in fact, that there were ideal numerical ratios which fix the proportions of the human body. Perfect art must follow these ideal proportions. He wrote a book, *The Canon,* on the subject and constructed the "Spear-bearer" to illustrate the thesis. These abstract types contrast sharply with what is found in numerous busts and statues of private individuals and military and political leaders made by Romans (Fig. 3–2).

Greek architecture also reveals the emphasis on ideal forms. The simple and austere buildings were always rectangular in shape; even the ratios of the dimensions employed were fixed. The Parthenon at Athens (Fig. 3–3) is an example of the style and proportions found in almost all Greek temples.