COURSE GUIDE

MATHEMATICS AND LOGIC:

FROM EUCLID TO MODERN GEOMETRY

A Brief History of Mathematics

Ancient Greeks like **Thales** and **Pythagoras** refocused the study of mathematics from concrete objects to abstractions and emphasized the importance of deductive reasoning to the attainment of new knowledge. As a result, the Greeks were able to make substantial strides in advancing the study of mathematics, and in particular geometry.

Euclid's *Elements* is perhaps the best example of the progress the ancient Greeks made. Composed by Euclid of Alexandria around 300 BC, the *Elements* took all that the Greeks knew about mathematics and compiled it into a rigorous and logically coherent system. Divided into thirteen books, the *Elements* begins with 23 definitions and ten axioms that serve as the foundations for the 465 propositions that Euclid proves. The *Elements* became the standard for mathematical education and went on to inform and influence almost all major mathematicians and scientists in the West for 2000 years.

19th century mathematicians, such as **Carl Friedrich Gauss (1777-1855)**, **János Bolyai (1802-1860)**, and **Nikolai Ivanovich Lobachevsky (1792-1856)**, set out independently to prove **Euclid's Fifth Postulate**. Although they each failed, their efforts produced the basis for a **non-Euclidean geometry**.

This non-Euclidean geometry, however, does not replace the system promulgated in the *Elements*, but validates and supports it. Non-Euclidean and Euclidean geometries are better understood as parallel geometries, each equally necessary, and each equally descriptive of reality.

Glossary of Logical Terms

Antecedent: In an implication, the conditional statement (i.e., the if-clause) is the antecedent. Whenever the if-clause precedes the then-clause in an implication, the antecedent is the first statement in the implication. For example, in the implication, *If P, then Q*, *P* is the antecedent. But, in the implication: "The game will be cancelled, if it rains outside," "if it rains outside," is the antecedent.

Consequent: In an implication, the result (i.e., the then-clause) that follows from the conditional is the consequent. Whenever the if-clause precedes the then-clause in an implication, the consequent is the second statement in the implication. For example, in the implication, *If P, then Q*, *Q* is the consequent. But, in the implication, "the game will be cancelled, if it rains outside," "the game will be cancelled," is the consequent.

Contrapositive: The contrapositive of an implication negates both antecedent and consequent and switches them. Thus, in the implication *If P, then Q*, the contrapositive states *If not Q, then not P*. The contrapositive of the implication, "If it is a bird, then it flies," is, "If it does not fly, then it is not a bird." Note that the contrapositive is logically equivalent to the original implication. We can see that in this example because, all birds fly.

Converse: The converse of an implication switches the antecedent and the consequent. Thus, in the implication *If P, then Q*, the converse states *If Q, then P*. For example, the converse of the implication, "If it is a bird, then it flies," is "If it flies, then it is a bird." Note that the converse is not logically equivalent with the original implication. In this example, not all things that fly are birds. Ladybugs, bats, and planes can all fly, but they are not birds.

Deductive reasoning: The process of reasoning through premises to reach a conclusion that necessarily follows from those premises. If the argument follows the rules of deductive logic, the argument is 'valid.' If the premises are true, the argument is 'sound.' When a deductive argument is both valid, and true, then the conclusion is necessarily true, and this can be known with absolute certainty. A syllogism is an example of deductive reasoning. To discredit the conclusion of a valid syllogism, at least one of the premises must be shown to be false.

Direct proof: An argument that begins by assuming a hypothesis is true and follows a logical progression of statements to reach a conclusion without assuming any more than is given in the premises. It proceeds step-by-step from those premises to the conclusion.

Implication: Implications are "if/then" statements made up of two simpler statements. For example, the two simple statements, "It is a bird," and "It flies," can be put together to form the implication, "If it is a bird, then it flies."

Indirect proof (*reductio ad absurdum*): An argument that not only assumes the given premises are true, but also assumes that its conclusion is false. The goal of such an argument is to demonstrate the impossibility and inconsistency of these two assumptions, thereby indirectly proving that the conclusion must actually be true. Also referred to as *reductio ad absurdum*.

Inverse: The inverse of an implication negates both the antecedent and consequent. Thus, in the implication *If P, then Q*, the inverse states *If not P, then not Q*. The inverse of the implication, "If it is a bird, then it flies," is, "If it is not a bird, then it does not fly." Note that the inverse is not logically equivalent with the original implication. In this example, not all things that fly are birds.

***Modus ponens* (Affirming the antecedent):** The most basic type of syllogism, which reaches a conclusion by affirming the antecedent. The syllogism, "If Socrates is a man, then he is mortal. Socrates is a man. Therefore, Socrates is mortal," is a modus ponens because, to arrive at the conclusion, the argument affirms the antecedent – Socrates is a man.

***Modus tollens* (Denying the consequent):** Another basic type of syllogism, which reaches a conclusion by denying the consequent. The syllogism, "If Rex is a chicken, then he is a bird. Rex is not a bird. Therefore, Rex is not a chicken," is a modus tollens because, to arrive at the conclusion, the argument denies the consequent – Rex is a bird. *Modus tollens* is related to contrapositive.

Negation: A sentence containing the opposite truth value of a particular statement is its negation. If a statement is true, then its negation must be false, and if a statement is false, then its negation must be true. For example, the negation of the statement, “The hat is red,” is “The hat is not red.” And the negation of the statement, “The ball is not blue,” is, “The ball is blue.”

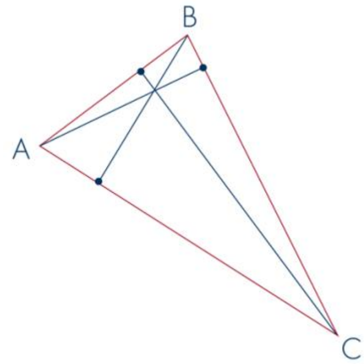
Statement: A declarative sentence that must be true or false. It cannot be both true and false, nor can it be neither true nor false. For example, the statement, “It is raining outside,” is either true or false depending on whether or not it is actually raining outside the moment the statement was uttered. The question, “Is it raining outside?” is not a statement because questions are neither true nor false.

Syllogism: A form of deductive reasoning consisting of at least a major premise, a minor premise, and a conclusion. In a syllogism the conclusion necessarily follows from the premises, and if the premises are true, then the conclusion must also be true. Syllogisms are often represented in a three-line form. For example,

All men are mortal. (Major Premise)
Socrates is a man. (Minor Premise)
Therefore, Socrates is mortal. (Conclusion)

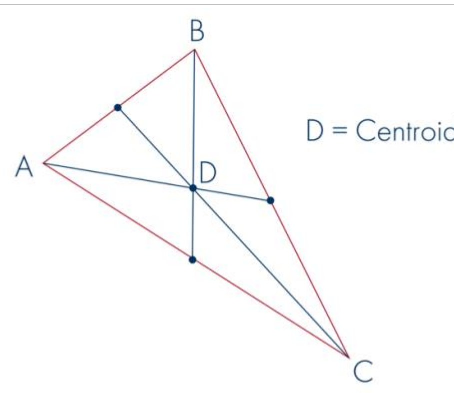
Glossary of Mathematical Terms

Altitude of a triangle: A line segment which runs through the vertex of a triangle and which lies perpendicular to the side opposite.

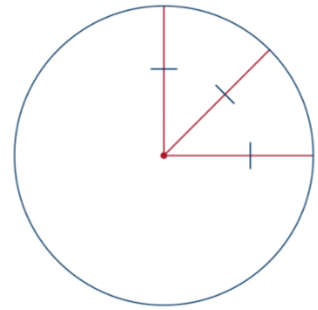


Axiom: In Euclidean geometry, axioms are statements we accept without proof, i.e., as self-evident truths.

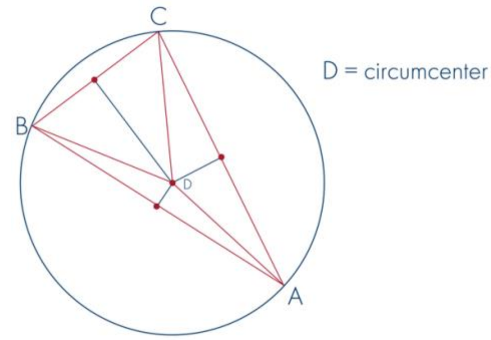
Centroid: One of four triangle centers in Euclidean geometry: Centroid, Circumcenter, Incenter, and Orthocenter. The centroid is the balancing point of the triangle, i.e., the point at which a physical triangle would balance perfectly in the air.



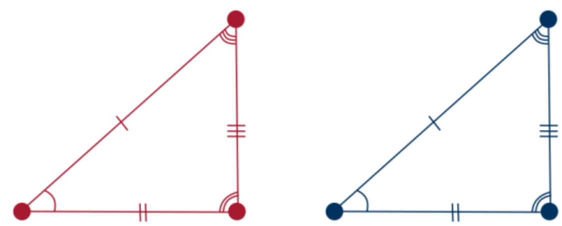
Circle: In Euclidean geometry, a circle is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure equal one another. That point is called the center of the circle.



Circumcenter: One of four triangle centers in Euclidean geometry: Centroid, Circumcenter, Incenter, and Orthocenter. The circumcenter lies at the center of the smallest circle that contains the triangle within it.



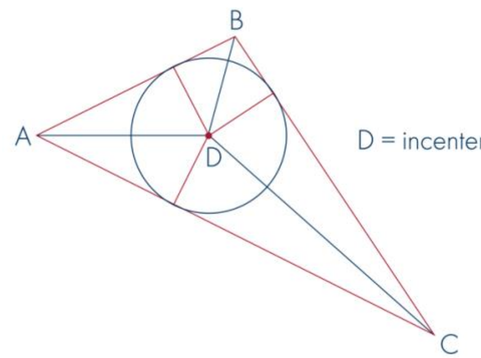
Congruence: Two things are congruent when they, and all their components, are identical. For example, two congruent triangles are two triangles equal in the length of their sides, in their angles, and in their areas.



Convex: A figure is convex when no line segment between two points inside the figure ever goes outside it. For example, a circle is convex.

Decomposition: The breaking down of a figure into its various parts.

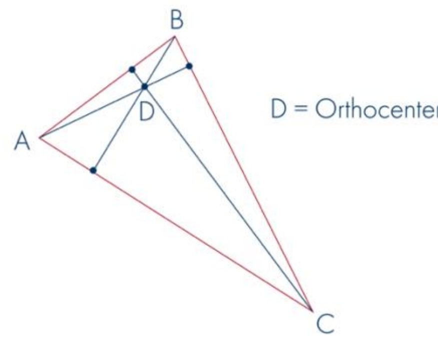
Incenter: One of four triangle centers in Euclidean geometry: Centroid, Circumcenter, Incenter, and Orthocenter. The incenter is the point at which the three interior angle bisectors meet and lies equidistant from the three sides of the triangle.



Irrational number: All real numbers that are not rational numbers, i.e., cannot be expressed as the ratio of two integers. For example, $\sqrt{2}$ is irrational.

Line: In Euclidean geometry, a line is length without breadth, the extremities of which are points.

Orthocenter: One of four triangle centers in Euclidean geometry: Centroid, Circumcenter, Incenter, and Orthocenter. The orthocenter is the point at which the three altitudes within a triangle intersect.



Parallel straight lines: In Euclidean geometry, parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

Point: In Euclidean geometry, a point is that which has no parts.

Prime number: Any number greater than 1 that is not a product of two smaller numbers. For example: 2, 3, 5, 7, and 11 are prime numbers.

Rational number: Any number that can be expressed as the ratio of two integers: a numerator and a non-zero denominator. For example, -3/7, or 5/1 are rational.

Right angle: In Euclidean geometry, a right angle occurs when one straight line is set up on another straight line and the corresponding angles are equal. See Euclid's postulates, IV.

Straight line: In Euclidean geometry, a straight line is a line that lies even with respect to the points on itself.

Superposition: The assumption that when a figure is moved from one location to another, the figure can be recreated exactly as before.

Euclid's Postulates

I: To draw a straight line from any point to any point.

II: To produce a finite straight line continuously in a straight line.

III: To describe a circle with any center and radius.

IV: That all rights angles equal one another.

V: That, if a straight line falling on two other straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Euclid's Common Notions

I: Things which equal the same thing also equal one another.

II: If equals are added to equals, the wholes are equal.

III: If equals are subtracted from equals, the remainders are equal.

IV: Things which coincide with one another equal one another.

V: The whole is greater than the part.