MATHEMATICS IN WESTERN CULTURE

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III

The Birth of the Mathematical Spirit

Whatever we Greeks receive, we improve and perfect.

PLATO

There is a story told of Thales that once during an evening walk he became absorbed in observation of the stars and fell into a ditch. A woman accompanying him exclaimed, 'How canst thou know what is doing in the heavens, when thou seest not what is at thy feet?' Thales, however, did do many things simultaneously and successfully. During one lifetime he not only founded Greek mathematics, observed the stars, and took nature walks with congenial companions, but also fathered Greek philosophy, contributed a major cosmological theory, traveled extensively, made notable contributions to astronomy, and realized enormous success in business.

Thales, along with most of the early Greek mathematicians, learned the elements of algebra and geometry from the Egyptians and the Babylonians. In fact, many of these scholars came from Asia Minor, which inherited the Babylonian culture. Others, born on the Greek mainland, went to Egypt and studied there. Despite the unquestioned influence of Egypt and Babylonia on Greek minds, the mathematics produced by the Greeks differed radically from that which preceded it. Indeed, from the point of view of the twentieth century, mathematics and, it may well be added, modern civilization began with the Greeks of the classical period, which lasted from about 600 to 300 B.C.

The mathematics that existed before Greek times has already been characterized as a collection of empirical conclusions. Its formulas were the accretion of ages of experiènce much as many medical practices and remedies are today. Though experience is no doubt a good teacher, in many situations it would be a most inefficient way of obtaining knowledge. Who would erect a mile-long bridge to determine whether a particular steel cable could support it? The method of trial and error may be direct but it may also be disastrous.

Is experience the only way of obtaining knowledge? Not for beings endowed with a reasoning faculty. Reasoning can follow many routes, among which is the commonly traveled one of analogy. The Egyptians, for example, believed in immortality and so they buried their dead with clothes, utensils, jewelry, and other things that might be of use in the next world. Their reasoning was that since life on Earth required these articles, the after-life would also.

Reasoning by analogy is useful, but it also has its limitations. There may not be an analogous situation at all; airplanes, radios, and submarines could hardly have been invented by reasoning by analogy. Or, there may be an analogous situation that differs slightly but enough to matter a great deal. Though human beings resemble apes, some conclusions about humans cannot be drawn from a study of the apes.

A more commonly used method of reasoning is known as induction. A farmer may observe that heavy rains during several successive springs were followed by excellent crops. He concludes that heavy rains are beneficial to crops. Again, because a person may have had unfortunate experiences in dealing with lawyers, he concludes that all lawyers are undesirable people. Essentially, the inductive process consists in concluding that something is *always* true on the basis of a limited number of instances.

Induction is the fundamental method of reasoning in experimental science. Suppose a scientist heats a given quantity of water from 40° to 70° and sees that the volume occupied by the water increases. If he is a good scientist, he will draw no conclusion as yet but will repeat the experiment many times. Let us suppose that he observes the same expansion each time. He will then declare that water expands as it is heated from 40° to 70°. This conclusion is obtained by inductive reasoning.

Though the conclusions obtained by inductive reasoning seem warranted by the facts, they are not established beyond all doubt. Logically these conclusions are not any better established than the generalization drawn from the observation of four hundred million Chinese that all human beings are yellow-skinned. In other words, we cannot be certain of any conclusion obtained by inductive reasoning. There are other limitations to this type of reasoning. We cannot conclude inductively what the effect on society of an untried law may be. Nor can we conclude inductively, as one uncritical observer did, that all Indians walk single file by seeing one do so!

The several methods of obtaining conclusions, each undoubtedly useful in a variety of situations, possess a common limitation: even if the facts of experience, or the facts on which reasoning by analogy or induction are based, are entirely correct, the conclusion obtained is not certain, and where certainty is vital these methods are practically useless.

Fortunately, there is a method of reasoning that does guarantee the certainty of the conclusions it produces. The method is known as deduction. Let us consider some examples. If we accept the facts that all apples are perishable and that the object before us is an apple, we *must* conclude that this object is perishable. As another example, if all good people are charitable and if I am good, then I must be charitable. And if I am not charitable I am not good. Again, we may argue deductively from the premises that all poets are intelligent and that no intelligent people deride mathematics, to the inevitable conclusion that no poet derides mathematics.

It does not matter, in so far as the reasoning is concerned, whether we agree with the premises. What is pertinent is that if we accept the premises we must accept the conclusion. Unfortunately, many people confuse the acceptability or truth of a conclusion with the validity of the reasoning that leads to this conclusion. From the premises that all intelligent beings are humans and that readers of this book are human beings, we might conclude that all readers of this book are intelligent. The conclusion is undoubtedly true but the purported deductive reasoning is invalid because the conclusion does not necessarily follow from the premises. A moment's reflection shows that even though all intelligent beings are humans there may be human beings who are not intelligent, and nothing in the premises tells us to which group of human beings the readers of this book belong.

Deductive reasoning, then, consists of those ways of deriving new statements from accepted facts that compel the acceptance of the derived statements. We shall not pursue at this point the question of why it is that we experience this mental conviction. What is important now is that man has this method of arriving at new conclusions and that these conclusions are unquestionable if the facts we start with are also unquestionable.

Deduction, as a method of obtaining conclusions, has many advantages over trial and error or reasoning by induction and analogy. The outstanding advantage is the one we have already mentioned, namely, that the conclusions are unquestionable if the premises are. Truth, if it can be obtained at all, must come from certainties and not from doubtful or approximate inferences. Second, in contrast to experimentation, deduction can be carried on without the use or loss of expensive equipment. Before the bridge is built and before the long-range gun is fired, deductive reasoning can be applied to decide the outcome. Sometimes deduction has the advantage of being the only available method. The calculation of astronomical distances cannot be carried out by applying a yardstick. Moreover, whereas experience confines us to tiny portions of time and space, deductive reasoning may range over countless universes and aeons.

With all of its advantages, deductive reasoning does not supersede experience, induction, or reasoning by analogy. It is true that 100 per cent certainty can be attached to the conclusions of deduction when the premises can be vouched for 100 per cent. But such unquestionable premises are not necessarily available. No one, unfortunately, has been able to vouchsafe the premises from which a cure for cancer could be deduced. For practical purposes, moreover, the certainty deduction grants is sometimes superfluous. A high degree of probability may suffice. For centuries the Egyptians used mathematical formulas drawn from experience. Had they waited for deductive proof the pyramids at Giza would not be squatting in the desert today.

Each of these various ways of obtaining knowledge, then, has its advantages and disadvantages. Despite this fact, the Greeks insisted that *all mathematical conclusions be established only by deductive reasoning*. By their insistence on this method, the Greeks were discarding all rules, formulas, and procedures that had been obtained by experience, induction, or any other non-deductive method and that had been accepted in the body of mathematics for thousands of years preceding their civilization. It would seem, then, that the Greeks were destroying rather than building; but let us withhold judgment for the present.

Why did the Greeks insist on the exclusive use of deductive proof in mathematics? Why abandon such expedient and fruitful ways of obtaining knowledge as induction, experience, and analogy? The answer can be found in the nature of their mentality and society.

The Greeks were gifted philosophers. Their love of reason and their delight in mental activity distinguished them from other peoples. The educated Athenians were as much devoted to philosophy as our smart-set is to night-clubbing; and pre-Christian fifth-century Athens was as deeply concerned with the problems of life and death, immortality, the nature of the soul, and the distinction between good and evil as twentieth-century America is with material progress. Philosophers do not reason, as do scientists, on the basis of personally conducted experimentation or observation. Rather their reasoning centers about abstract concepts and broad generalizations. It is difficult, after all, to experiment with souls in order to arrive at truths about them. The natural tool of philosophers is deductive reasoning, and hence the Greeks gave preference to this method when they turned to mathematics.

Philosophers are, moreover, concerned with truths, the few, immaterial wisps of eternity that can be sifted from the bewildering maze of experiences, observations, and sensations. Certainty is the indispensable element of truth. To the Greeks, therefore, the mathematical knowledge accumulated by the Egyptians and Babylonians was a house of sand. It crumbled to the touch. The Greeks sought a palace built of ageless, indestructible marble.

The Greek preference for deduction was, surprisingly, a facet of the Hellenic love for beauty. Just as the music lover hears music as structure, interval, and counterpoint, so the Greek saw beauty as order, consistency, completeness, and definiteness. Beauty was an intellectual as well as an emotional experience. Indeed, the Greek sought the rational element in every emotional experience. In a famous eulogy Pericles praises the Athenians who died in battle at Samos not merely because they were courageous and patriotic, but because reason sanctioned their deeds. To people who identified beauty and reason, deductive arguments naturally appealed because they are planned, consistent, and complete, while conviction in the conclusions offers the beauty of truth. It is no wonder, then, that the Greeks regarded mathematics as an art, as architecture is an art though its principles may be used to build warehouses.

Another explanation of the Greek preference for deduction is found in the organization of their society. The philosophers, mathematicians, and artists were members of the highest social class. This upper stratum either completely disdained commercial pursuits and manual work or regarded them as unfortunate necessities. Work injured the body and took time from intellectual and social activities and the duties of citizenship.

Famous Greeks spoke out unequivocally about their disdain of work and business. The Pythagoreans, an influential school of philosophers and religionists we shall soon meet, boasted that they had raised arithmetic, the tool of commerce, above the needs of merchants. They sought knowledge, not wealth. Arithmetic, said Plato, should be pursued for knowledge and not for trade. Moreover, he declared the trade of a shopkeeper to be a degradation for a freeman and wished the pursuit of it to be punished as a crime. Aristotle declared that in a perfect state no citizen should practice any mechanical art. Even Archimedes, who contributed extraordinary practical inventions, cherished his discoveries in pure science and considered every kind of skill connected with daily needs ignoble and vulgar. Among the Boeotians there was a very decided contempt for work. Those who defiled themselves with commerce were excluded from state office for ten years.

The Greek attitude toward work might have had little influence on their culture were it not for the fact that they did possess a large slave class to whom they could 'pass the buck.' Slaves ran the businesses and the households, did unskilled and technical work, managed the industries, and practiced even the most important professions such as medicine. The slave basis of classical Greek society fostered a divorce of theory from practice and the development of the speculative and abstract side of science and mathematics with a consequent neglect of experimentation and practical applications.

In view of the eschewal of commerce and trade by the Greek upper class—certainly a contrast to the preoccupation of our highest social class with finance and industry—it is not hard to understand the preference for deduction. If a person does not 'live' in the world about him, experience teaches him very little. Similarly, in order to reason inductively or by analogy he must be willing to go about and observe the real world. Experimentation would certainly be alien to thinkers who frowned upon the use of the hands. Since the Greeks were not idlers they fell quite naturally into the mode of inquiry that suited their tastes and social attitudes.

Jonathan Swift observed and ridiculed this isolation of Greek culture, as well as its influence on the abstract nature of what he believed to be the pseudo-science of his own day. When Gulliver is led on a tour of inspection of Laputa, he observes:

> Their houses are very ill built, the walls bevil, without one right angle in any apartment, and this defect ariseth from the contempt they bear to practical geometry, which they despise as vulgar and mechanic, those instructions they give being too refined for the intellectuals of their workmen, which occasions perpetual mistakes. And although they are dexterous enough upon a piece of paper in the management of the rule, the pencil, and the divider, yet in the common actions and behaviour of life, I have never seen a more clumsy, awkward, and unhandy people, nor so slow and perplexed in their conceptions upon all other subjects, except those of mathematics and music.

Nevertheless, Greek insistence on deductive reasoning as the sole method of proof in mathematics was a contribution of the first magnitude. It removed mathematics from the carpenter's tool box, the farmer's shed, and the surveyor's kit, and installed it as a system of thought in man's mind. Man's reason, not his senses, was to decide thenceforth what was correct. By this very decision reason effected an entrance into Western civilization, and thus the Greeks revealed more clearly than in any other manner the supreme importance they attached to the rational powers of man.

The exclusive use of deduction has, moreover, been the source of the surprising power of mathematics and has differentiated that subject from all other fields of knowledge. In particular, therein lies one sharp distinction between mathematics and science, for science also uses conclusions obtained by experimentation and induction. Consequently, the conclusions of science occasionally need revision and sometimes must be thrown overboard entirely, whereas the conclusions of mathematics have stood for thousands of years even though the reasoning in some cases has had to be supplemented.

Had the Greeks done no more to the character of mathematics than to convert it from an empirical science into a deductive system of thought their influence on history would still have been enormous. But their contributions only began there.

A second vital contribution of the Greeks consisted in their having *made mathematics abstract.* Earlier civilizations learned to think about numbers and operations with numbers somewhat abstractly, but only in the unconscious manner in which we as children learned to think about and manipulate them. Geometrical thinking, before Greek times, was even less advanced. To the Egyptians, for example, a straight line was quite literally no more than either a stretched rope or a line traced in sand. A rectangle was a fence bounding a field.

With the Greeks not only was the concept of number consciously recognized but also they developed *arithmetica,* the higher arithmetic or theory of numbers; at the same time mere computation, which they called *logistica* and which involved hardly any appreciation of abstractions, was deprecated as a skill in much the same way as we look down upon typing today. Similarly in geometry, the words *point, line, triangle,* and the like became mental concepts merely suggested by physical objects but differing from them as the concept of wealth differs from land, buildings, and jewelry and as the concept of time differs from a measure of the passage of the sun across the sky.

The Greeks eliminated the physical substance from mathematical concepts and left mere husks. They removed the Cheshire cat and left the grin. Why did they do it? Surely it is far more difficult to think about abstractions than about concrete things. One advantage is immediately apparent—the gain in generality. A theorem proved about the abstract triangle applies to the figure formed by three match sticks, the triangular boundary of a piece of land, and the triangle formed by the Earth, sun, and moon at any instant.

The Greeks preferred the abstract concept because it was, to them, permanent, ideal, and perfect whereas physical objects are short-lived, imperfect, and corruptible. The physical world was unimportant except in so far as it suggested an ideal one; man was more important than men. The strong preference for abstractions will be evident from a brief glance at the leading doctrine of Greece's greatest philosopher.

Plato was born in Athens about 428 B.C. of a distinguished and active Greek family, at a time when that city was at the height of her power. While still a youth he met Socrates and later supported him in the defense of the aristocracy's leadership of Athens. When the democratic party took power, Socrates was sentenced to drink poison and Plato became *persona non grata* in Athens. Convinced that there was no place in politics for a man of conscience—of course, politics was different in those days—he decided to leave the city. After traveling extensively in Egypt and visiting the Pythagoreans in lower Italy, he returned to Athens about 387 B.C. where he founded his academy for philosophy and scientific research. Plato devoted the latter forty of his eighty years of life to teaching, writing, and the making of mathematicians. His pupils, friends, and followers were the greatest men of his age and of many succeeding generations, and among them could be found every noteworthy mathematician of the fourth century B.C.

There is, Plato maintained, the world of matter, the Earth and the objects on it, which we perceive through our senses. There is also the world of spirit, of divine manifestations, and of ideas such as Beauty, Justice, Intelligence, Goodness, Perfection, and the State. These abstractions were to Plato as the Godhead is to the mystic, the Nirvana to the Buddhist, and the spirit of God to the Christian. Whereas our senses grasp the passing and the concrete, only the mind can attain the contemplation of these eternal ideas. It is the duty of every intelligent man to use his mind toward this end, for these ideas alone, and not the daily affairs of man, are worthy of attention. These idealizations, which are the core of Plato's philosophy, are on exactly the same mental level as the abstract concepts of mathematics. To learn how to think about the one is to learn how to think about the other. Plato seized upon this relationship.

In order to pass from a knowledge of the world of matter to the world of ideas, he said, man must prepare himself. Light from the highest realities, which reside in the divine sphere, blinds the person who is not trained to face it. He is, to use Plato's own famous figure, like one who lives continually in the deep shadows of a cave and is suddenly brought out into the sunlight. To make the transition from darkness to light, mathematics is the ideal means. On the one hand, it belongs to the world of the senses, for mathematical knowledge pertains to objects on this Earth. It is, after all, the representation of properties of matter. On the other hand, considered solely as idealization, solely as an intellectual pursuit, mathematics is indeed distinct from the physical objects it describes. Moreover, in the making of proofs, physical meanings must be shut out. Hence mathematical thinking prepares the mind to consider higher forms of thought. It purifies the mind by drawing it away from the contemplation of the sensible and perishable to the eternal. The path to salvation, then, to the understanding of Truth, Beauty, and Goodness, led through mathematics. This study was an initiation into the Mind of God. In Plato's words, '. . . geometry will draw the soul towards truth, and create the spirit of philosophy, . . .' For geometry is concerned not with material things but with points, lines, triangles, squares, and so on, as objects of pure thought.

Arithmetic, too, said Plato, 'has a very great and elevating effect, compelling the soul to reason about abstract numbers, and rebelling against the introduction of visible or tangible objects into the argument.' He advised 'the principal men of our State to go and learn arithmetic, not as amateurs, but they must carry on the study until they see the nature of numbers with the mind only.'

To sum up Plato's position: a modicum of geometry and calculation suffice for practical needs; however, the higher and more advanced portions tend to lift the mind above mundane considerations and enable it to apprehend the final aim of philosophy, the idea of the Good. For this reason Plato recommended that the future philosopher-kings be trained for ten years, from the age of twenty to the age of thirty, in the study of the exact sciences: arithmetic, plane geometry, solid geometry, astronomy, and harmonics. In his stress on mathematics as a preparation for philosophy, Plato spoke not merely for his followers and for his generation but for the whole classical Greek age.

The Greek preference for idealizations and abstractions expressed itself in philosophy and mathematics. It showed itself just as clearly in art. Greek sculpture of the classical period dwelt not on particular men and women but on ideal types (plates I and II). This idealization extended to standardization of the ratios of the parts of the body to each other. No finger or toenail was overlooked in Polyclitus' prescriptions of these ratios. The modern practice in beauty contests of awarding the prize to the girl whose measurements most closely approximate an established standard is a continuation of the Greek interest in an ideal figure.

The faces and postures of the classical Greek draped and undraped figures, at least until the decadent 'Laocoön,' show no emotion or concern. Judged by their facial expressions the Greek gods and the Greek people neither thought, nor laughed, nor worried. Their demeanor is calm even in pieces of sculpture depicting dramatic action.

The faces are as serene as we could expect those of man in the abstract to be. Particular emotions are, after all, a matter of the moment, whereas these sculptors were depicting the eternal in the nature of man. This epic style of sculpture contrasts sharply with what is found in the numerous busts and statues of military and political leaders done in the Roman period (plate III).

The Greeks standardized their architecture as they did their sculpture. Their simple and austere buildings were always rectangular in shape; even the ratios of the dimensions were fixed. The Parthenon at Athens (plate IV) is an example of the style and proportions found in almost all Greek temples. The insistence on ideal dimensions is, incidentally, closely related to the Greek insistence on form, form in the abstract, a concept not alien to our day, in which art and abstraction are practically synonymous.

The insistence on deductive and abstract mathematics created the subject as we know it. Both of these characteristics were imparted by philosophers. Despite the fact that mathematics was born of Greek philosophy, many great mathematicians and some of the not so great have been extremely scornful of all philosophic speculation. Of course this attitude is no more than an expression of narrowness. These mathematicians are in their chosen field like mighty rivers that wear down mountains to reach the sea but whose paths are then confined to narrow gorges. Their power has enabled them to penetrate deeply below the surfaces they started to explore but has also enclosed and entrapped them in high walls over which they can no longer see. These disdainful mathematicians overlook the fact that the deepest and mightiest rivers are continually fed by tenuous, vaguely defined clouds. So, too, do the clouds of philosophic thought distill their essence into mathematical streams.

The Greeks put their stamp on mathematics in still another way that has had a marked effect on its development, namely, by their emphasis on geometry. Plane and solid geometry were thoroughly explored. A convenient method of representing quantities, however, was never developed nor were efficient methods of reckoning with numbers. Indeed, in computational work they even failed to utilize techniques the Babylonians had created. Algebra in our present sense of a highly efficient symbolism and numerous established procedures for the solution of problems was not even envisioned. So marked was this disparity of emphasis that we are impelled to seek the reasons for it. There are several.

We mentioned earlier that in the classical period industry, commerce, and finance were conducted by slaves. Hence the educated people, who might have produced new ideas and new methods for handling numbers, did not concern themselves with such problems. Why worry about the use of numbers in measurement if one doesn't measure, or in trading if one dislikes trade? Nor do philosophers need the numerical dimensions of even one rectangle to speculate about the properties of all rectangles.

Like most philosophers the Greeks were star-gazers. They studied the heavens to penetrate the mysteries of the universe. But the use of astronomy in navigation and calendar reckoning hardly concerned the Greeks of the classical period. For their purposes, shapes and forms were more relevant than measurements and calculations, and so geometry was favored. Of these forms, the circle and sphere, suggested of course by superficial observation of the sun, moon, and planets, received the major share of attention. Hence their astronomical interests, too, led the classical Greeks to favor geometry.

The twentieth century seeks reality by breaking matter down—witness our atomic theories. The Greeks preferred to build matter up. For Aristotle and other Greek philosophers the form of an object is the reality to be found in it. Matter as such is primitive and shapeless; it is significant only when it has shape. It is no wonder, then, that geometry, the study of forms, was the special concern of the Greeks.

Finally, it was the solution of a vital mathematical problem that drove the Greek mathematicians into the camp of the geometers. We have already spoken of the fact that the Babylonian civilization, as well as earlier ones, used integers and fractions. The Babylonians were familiar also with a third type of number which arose through the application of a theorem on right triangles.

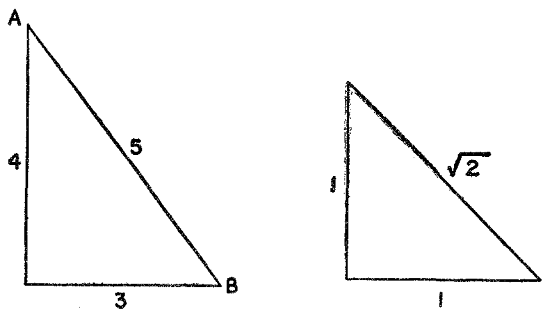
First, let us examine the theorem. If a right triangle has arms of lengths 3 and 4, the hypotenuse, or side opposite the right angle (AB in fig. 2) has length 5. Now the square of 5, namely 25, is the sum of the squares of 3 and 4, i.e., $5^2 = 3^2 + 4^2$. This relationship among the sides of a right triangle, that is, that the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the other two sides, is commonly known as the Pythagorean theorem.

To the Babylonians and Egyptians the fact, if not a proof, of this relationship was known.

Suppose now that the arms of a right triangle both have length 1 (fig. 3). What would be the length of the hypotenuse? Let us call the hypotenuse x. Then according to the Pythagorean theorem its length must be such that

$$x^2 = 1^2 + 1^2 = 2.$$

Hence x, the length of the hypotenuse, must be a number whose square is 2. We indicate the number whose square is 2 by $\sqrt{2}$ and



Figures 2 and 3. Two right triangles

call it the square root of 2. But what number equals $\sqrt{2}$? That is, what number multiplied by itself gives 2?

The answer, as the Pythagorean school of mathematicians discovered to its great dismay, is that there is no whole number or fraction whose square is 2. $\sqrt{2}$ is a new kind of number, and they called it *irrational* because it could not be expressed exactly as a ratio of whole numbers, as 4/3 or 3/2. By contrast, whole numbers and fractions are called *rational* numbers. These terms are in use today.

The irrational number is a much neglected topic in the history of thought and a troublesome member of our number system. We have just seen that such numbers must be used in order to represent lengths and they are, moreover, explicitly and implicitly involved in almost all of mathematics. Yet how can we add, subtract, multiply, or divide such numbers? For example, how can we add 2 and $\sqrt{2}$? How do we divide $\sqrt{7}$ by $\sqrt{2}$?

The Babylonians had a makeshift, though practical, solution of these difficulties. They approximated the value of $\sqrt{2}$. For example, since the square of 14/10 or 1.4 is 1.96, and since 1.96 is nearly equal to 2, 1.4 must be nearly equal to $\sqrt{2}$. An even better approximation to $\sqrt{2}$ is 1.41 because the square of 1.41 is 1.988.

The Babylonian approximation to $\sqrt{2}$ does not permit exact reasoning with irrational numbers, for no matter how many decimal places we are willing to use we cannot write a rational number whose square is *exactly* 2. Yet, if mathematics is to merit its claim to being an exact study, it must evolve a method of working with $\sqrt{2}$ itself and not an approximation of it. To the Greek mind, this difficulty was as genuine and as prepossessing as the problem of food to a castaway on a coral reef.

Not content to use the less scrupulous method of the Babylonians, the Greeks undertook to face the logical difficulty squarely. In order to think about irrational numbers with exactness they conceived the idea of working with all numbers geometrically. They started out this way. A length was chosen to represent the number 1. Other numbers were then represented in terms of this length. To represent $\sqrt{2}$, for example, they used a length equal to the hypotenuse of a right triangle whose sides were one unit in length. The sum of 1 and $\sqrt{2}$ was a length formed by adjoining a unit segment to the length representing $\sqrt{2}$. In this geometrical form the sum of a whole number and an irrational one is no more difficult to conceive than the sum of one and one.

Similarly the product of two numbers, 3 and 5 for example, was expressed geometrically as the area of the rectangle with dimensions 3 and 5. In the case of 3 and 5 the use of area as a way of thinking about the product may be no great advantage. But one can also think of the product of 3 and $\sqrt{2}$ as an area. To think about this second rectangle is no more difficult than to think of the first one; yet it provides an exact way of working with the product of an integer and an irrational number or, for that matter, two irrational numbers.

The Greeks not only operated with numbers in the geometric manner but went so far as to solve equations involving unknowns by series of geometrical constructions. The answers to these constructions were line segments whose lengths were the unknown values. The thoroughness of their conversion to geometry may be judged from the fact that the product of four numbers was unthinkable in classical Greece because there was no geometric figure to represent it in the manner that area and volume represented the product of two and three numbers respectively. Incidentally, we still speak of a number such as 25 as the *square* of 5 and of 27 as the *cube* of 3 in conformity with Greek thought.

The preference of the Greeks for geometry was so marked that during his travel in Laputa, Gulliver was again forced to comment:

> The knowledge I had in mathematics gave me great assistance in acquiring their phraseology, which depended much upon that science and music; and in the latter I was not unskilled. Their ideas are perpetually conversant in lines and figures. If they would, for example, praise the beauty of a woman, or any other animal, they describe it by rhombs, circles, parallelograms, ellipses, and other geometrical terms, or by words of art drawn from music, needless here to repeat. I observed in the King's kitchen all sorts of mathematical and musical instruments, after the figures of which they cut up the joints that were served to his Majesty's table.

Because the Greeks converted arithmetical ideas into geometrical ones and because they devoted themselves to the study of geometry, that subject dominated mathematics until the nineteenth century, when the difficulties in treating irrational numbers on an exact, purely arithmetical basis were finally resolved. In view of the clumsiness and complexity of arithmetical operations geometrically performed, this conversion was, from a practical standpoint, a highly unfortunate one. The Greeks not only failed to develop the number system and algebra which industry, commerce, finance, and science must have, but they also hindered the progress of later generations by influencing them to adopt the more awkward geometrical approach. Europeans became so habituated to Greek forms and fashions that Western civilization had to wait for the Arabs to bring a number system from far-off India.

Unfortunate as this Greek perversion of the number system and of algebra may appear to us with our understanding of progress, it still should not invoke on Greek heads the condemnation that has sometimes been heaped there. The one backward step the Greeks took was in itself thoroughly reasonable; moreover, the damage done is heavily outweighed by the incomparable good of their other accomplishments.

When most people describe the Greek contributions to modern civilization, they talk in terms of art, philosophy, and literature. No doubt the Greeks deserve the highest praise for what they bequeathed to us in these fields. Greek philosophy is as alive and significant today as it was then. Greek architecture and sculpture, especially the latter, are more beautiful to the average educated person of the twentieth century than the creations of his own age. Greek plays still appear on Broadway. Nevertheless, the contribution of the Greeks that did most to determine the character of present-day civilization was their mathematics. By altering the nature of the subject in the manner we have related, they were able to proffer their supreme gift. This we proceed to examine.